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Binomial Theorem

Revise Notes

Binomial Theorem

We know how to find the squares and cubes of binomials like a + b and a – b. E.g. $(a+b)^2$, $(a-b)^3$ etc. However, for higher powers calculation becomes difficult. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number.

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_{n-1} a.b^{n-1} + {}^nC_n b^n$$

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k$$

Points to note in Binomial Theorem:

Total number of terms in expansion = index count +1. g. expansion of $(a + b)^2$, has 3 terms.

Powers of the first quantity ‘a’ go on decreasing by 1 whereas the powers of the second quantity ‘b’ increase by 1, in the successive terms.

In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of a + b.

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Numerical:

Compute $(98)^5$

Solution:

$(98)^5 = (100-2)^5 = {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2 - {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$

$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$

$= 10040008000 - 1000800032 = 9039207968$

General Term

General term in the expansion of $(a+b)^n$ is $_{Tr+1} = {}^nC_r\; a^{n-r}\, b^r$

Numerical:

Find the 4th term in the expansion of (x – 2y)12

Solution:

Putting r=3,n=12, a =x & b= -2y in this formula the formula,

$T_{r+1} = {}^nC_r\; a^{n-r}\, b^r$

$T_4 = {}^{12}C_3\,(x)^9(-2y)^3$

Or $T_4 = -1760\, x^9\, y^3$

Middle Term

Middle term in the expansion of $(a+b)^n$

Case 1: "n" is even, Total term is expansion : n+1 à Odd

Middle term = $(\frac{n}{2}+1)^{th}$

We can now find the middle term using the general term formula,

$T_{r+1} = {}^nC_r\ a^{n-r}\, b^r$

Case 2: "n" is odd, total term in expansion: n+1 à Even

There will be two Middle terms = $(\frac{n+1}{2})^{th}$ & $(\frac{n+1}{2}+1)^{th}$ terms

We can now find the middle term using the general term formula

$T_{r+1} = {}^nC_r\ a^{n-r}\, b^r$

Numerical:

Find the middle terms in the expansions of $(\frac{x}{3}+9y)^{10}$

Solution:

In this case, $a=\frac{x}{3}$, b=9y & n = 10

Since n=10, number of terms in the expansion will be 11.

Middle terms will be $\frac{11+1}{2}$ = 6

To find T_6, we need to put r=5 in the formula

$T_{r+1} = {}^nC_r\ a^{n-r}\, b^r$

$T_{5+1} = {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5$

Or $T_6 = 61236x^5y^5$

Greatest Coefficient and Term:

Greatest Coefficient = $\begin{cases} {}^nC_{n/2}, & \text{if n is even} \\ {}^nC_{(n-1)/2} \text{ and } {}^nC_{(n+1)/2}, & \text{if n is odd} \end{cases}$

To find the numerically greatest term in the expansion of $(x + a)^n$:

Calculate, m = $\frac{(n+1)}{1+\left|\frac{a}{x}\right|}$

Case 1: If m ∈ Integer, then T_m and T_{m+1} are the greatest terms and both are numerically equal.

Case 2: If m ∉ Integer, then $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integer function.

Numerical:

Find numerically greatest term in the expansion of $(2 + 3x)^8$, when x = $\frac{2}{3}$.

Solution:

Here, m = $\frac{(8+1)}{1+\left|\frac{3\times\frac{2}{3}}{2}\right|} = \frac{9}{2} = 4.5$

Since 4.5 is an integer, so $T_{[4.5]+1} = T_5$ is numerically greatest term.

$T_5 = {}^8C_4(2)^4(3x)^4 = {}^8C_4(2)^4\left(3 \times \frac{2}{3}\right)^4 = {}^8C_4(2)^4(2)^4 = {}^8C_4(2)^8$

Divisibility and Remainder:

Divisibility:

(i) $(x^n - a^n)$ is divisible by (x - a), ∀ n ∈ N.

(ii) $(x^n + a^n)$ is divisible by (x + a), ∀ n ∈ Only odd natural numbers.

Finding Remainder using Binomial Theorem:

If a, p, n and r positive integers, then to find the remainder when a^{pn+r} is divided by b, we adjust power of a to a^{pn+r} which is very close to b, say with difference 2 i.e. $b \pm 2$.

Also, the remainder is always positive. When number of the type 5n – 2 is divided by 5, then we have

5) 5n – 2 (n

5n

———
-2

We can write – 2 = - 2 – 3 + 3 = - 5 + 3

Or $\frac{5n-2}{5} = \frac{5n-5+3}{5} = n - 1 + \frac{3}{5}$

Hence, the remainder is 3.

Important Results:

(i) $2 \leq \left(1 + \frac{1}{n}\right)^n < 3, n \geq 1, n \in N$

(ii) If n > 6, then $\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$

Numerical:

If 7^{103} is divided by 25, find the remainder.

Solution:

We have, $7^{103} = 7.\ 7^{102} = 7.(7^2)^{51} = 7.(49)^{51} = 7.(50-1)^{51}$

$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \cdots - 1]$

$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \cdots + {}^{51}C_{50}(50)] - 7 - 18 + 18$

$= 7[50k] - 25 + 18$, where k is an integer.

$= 25[14k - 1] + 18$

$= 25p + 18$ [where p is an integer]

Now, $\frac{7^{103}}{25} = p + \frac{18}{25}$.

Hence, the remainder is 18.

Sum of Binomial Coefficients

In the binomial expansion of $(1+x)^n$, let us denote the coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_r$, ..., ${}^nC_n$ by C_0, C_1, C_2, ..., C_r, ..., C_n, respectively.

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$.

Putting x = 1, we get:

$\mathbf{C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n}$

Putting x = - 1, we get:

$C_0 - C_1 + C_2 - C_3 + C_4 - C_5 + \ldots = 0$

$\mathbf{C_1 + C_3 + C_5 + \ldots = C_0 + C_2 + C_4 + \ldots = 2^{n-1}}$

Use of Differentiation:

This method is applied only when numerical occur as the product of the binomial coefficients.$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$.

Solution:

$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$.

(i) If last term of the series leaving the plus or minus sign is m, then divide m by n. If q is quotient and r is the remainder.

Then replace x by x^q in the given series and multiplying both sides of the expression by x^r.

(ii) Now, differentiate both sides w.r.t. x and put x = 1 or – 1 or i, etc according the given series.

(iii) If the product of two numerical or three numerical, then differentiate twice or thrice accordingly.

Use of Integration:

This method is applied only when the numerical occur as the denominator of the binomial coefficient.

Solution:

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$.

Integrate both sides between suitable limits which gives the required series.

(i) If the sum contains $C_0, C_1, C_2, C_3, \dots, C_n$ with all positive signs, then integrate between limits 0 to 1.

(ii) If the sum contains alternate signs (i.e. +, -), then integrate between limits – 1 and 0.

Numerical:

Prove that $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n.\,2^{n-1}$.

Solution:

Here, last term of $C_1 + 2C_2 + 3C_3 + \dots + nC_n$ is nC_n i.e. n and with positive sign.

Then, n = n.1 + 0

Here, q = 1 and r = 0.

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$

Differentiating both sides w.r.t. x, we get

$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$

Putting x = 1, we get:

$n.\,2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$

or $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.\,2^{n-1}$

Numerical:

Prove that $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

Solution:

We have $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Integrating both the sides within limits 0 to 1, then we get:

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n)\,dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1}$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Binomial Expansion for Negative and Fractional Indices

For all $n \in Q$ (Rational Numbers)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

Numerical:

Write the expansion for $(1-x)^{-1}$.

Solution:

$(1-x)^{-1} = 1 + (-1)(-x) + \frac{-1(-1-1)}{2!}(-x)^2 + \frac{-1(-1-1)(-1-2)}{3!}(-x)^3 + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

Numerical:

Find the third term in the expansion of $(1+2x)^{1/2}$.

Solution:

We know that $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots$

Now, $T_3 = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(2x)^2$

$T_3 = -\frac{1}{2}(x)^2$

Multinomial Theorem

If n is a positive integer and $x_1, x_2, x_3, \ldots, x_k \in C$, then

$$(x_1 + x_2 + x_3 + \ldots + x_k)^n = \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are all non-negative integers such that

$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n$.

Coefficient of ${x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3} \ldots {x_k}^{\alpha_k}$ in the expansion of

$(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is $\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)}$

Numerical:

Find the coefficient of $w^4x^3y^2z$ in the expansion of $(w - x + y - z)^{10}$.

Solution:

The coefficient of $w^4x^3y^2z$ in the expansion of $(w - x + y - z)^{10}$ is

$(-1)^4 \frac{10!}{4!3!2!1!} = 12600$

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